COMPLEXITY OF GRAPHS WITH WHEEL GRAPH AND FAN GRAPH AS THEIR BLOCKS

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FINAL YEAR PROJECT

DEPARTMENT OF MATHEMATICS
FACULTY OF MATHEMATICS AND NATURAL SCIENCES
UNIVERSITAS TANJUNGPURA
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as one of the requirements to obtain a degree
in Bachelor of Mathematics at the Department of Mathematics

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Kompleksitas Graf dengan Blok Bentuk Graf Roda dan Graf Kipas

Abstrak

Diberikan sebarang graf G yang terhubung dan tak berarah. Suatu graf terhubung disebut *nonseparable graph* jika tidak memuat simpul pemutus. Blok dari graf G adalah subgraf maksimal yang *nonseparable* dari graf G. Setiap graf terhubung G memuat pohon perentang, yaitu subgraf dari graf G yang berupa pohon dan memuat semua simpul dari G. Kompleksitas graf G merupakan banyaknya pohon perentang graf G, dinotasikan dengan $\tau(G)$. Teorema pohon matriks menyatakan banyaknya pohon perentang graf G sama dengan nilai sebarang kofaktor dari matriks Laplacian $L(G)$ dimana $L(G)$ menyatakan selisih antara matriks *degree* $D(G)$ dengan matriks *adjacency* $A(G)$. Penelitian ini bertujuan untuk merumuskan kompleksitas graf dengan blok bentuk graf roda W_n dan graf kipas F_n. Penggunaan graf pada penelitian ini dibatasi untuk graf yang memiliki karakteristik tertentu yaitu graf yang memuat blok bentuk graf W_n dan graf F_n. Penelitian ini diawali dengan menentukan matriks *adjacency* dan matriks *degree*. Kemudian, mencari matriks Laplacian. Selanjutnya, menghitung nilai sebarang kofaktor dari matriks Laplacian. Akhirnya, merumuskan formula kompleksitas graf sehingga diperoleh formula kompleksitas masing-masing graf dan dibuktikan kebenarannya. Hasil dari penelitian ini adalah $\tau(W_n) = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2$, $\tau(F_n) = \frac{1}{5}\sqrt{5}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right]$, dan beberapa persamaan yang menghubungkan $\tau(W_n)$ dan $\tau(F_n)$ dengan graf yang memuat blok bentuk graf W_n dan graf F_n.

Kata Kunci: pohon perentang, teorema pohon matriks, matriks Laplacian.

Complexity of Graphs with Wheel Graph and Fan Graph as their Blocks

Abstract

Let G be a connected and undirected graph. A connected graph is said to be a nonseparable graph if it has no cut vertices. A block of graph G is a maximal nonseparable subgraph of G. Every connected graph G has a spanning tree which is a subgraph of G that is a tree and contains all vertices of G. The complexity of a graph G is the number of spanning trees of G, denoted by $\tau(G)$. The matrix tree theorem states that the number of spanning trees of a graph G equals the value of any cofactor of the Laplacian matrix where $L(G)$ is the difference between the degree matrix $D(G)$ and the adjacency matrix $A(G)$. This study aims to formulate the complexity formula of graphs with blocks in the form of wheel graph W_n and fan graph F_n. In this study, the graphs are limited to graphs that have specific characteristics, namely graphs that contain blocks in the form of W_n and F_n. The study begins by determining the adjacency matrix and degree matrix. Then, find the Laplacian matrix. Next, compute the value of any cofactor of the Laplacian matrix. Finally, formulate the complexity formula of graph so that the complexity formula of each graph is obtained and its validity is proven. The results of this study are $\tau(W_n) = \left(\frac{3+\sqrt{5}}{2}\right)^n + \left(\frac{3-\sqrt{5}}{2}\right)^n - 2$, $\tau(F_n) = \frac{1}{5}\sqrt{5}\left[\left(\frac{3+\sqrt{5}}{2}\right)^n - \left(\frac{3-\sqrt{5}}{2}\right)^n\right]$, and several equations relating $\tau(W_n)$ and $\tau(F_n)$ to graphs containing blocks in the form of W_n and F_n.

Keywords: spanning tree, matrix tree theorem, Laplacian matrix.

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The author is aware that this final year project still needs some suggestions, input, and constructive criticism. It is hoped that this final year project can make a significant contribution to the development of knowledge, particularly in the field of mathematics, and be beneficial for readers and utilized optimally.

Pontianak, June 2nd, 2025

Alexander

CONTENTS

APPROVAL SHEET ii
ACADEMIC INTEGRITY STATEMENT iii
Abstrak iv
Abstract v
ACKNOWLEDGEMENTS vi
CONTENTS vii
LIST OF FIGURES ix
LIST OF TABLES x
LIST OF SYMBOLS xi
CHAPTER I INTRODUCTION 1
1.1 Research Background 1
1.2 Problem Statements 2
1.3 Research Aims 2
1.4 The Scope of the Study 2
1.5 Literature Review 3
1.6 Research Methodology 4
CHAPTER II THEORETICAL BACKGROUND 6
2.1 Basic Concepts of Graphs 6
2.2 Trees 8
2.3 Types of Graphs 9
2.4 Matrices dan Determinants 14
2.5 Matrix Representations of Graphs 23
2.6 Complexity of Graphs 32
CHAPTER III COMPLEXITY OF SEVERAL GRAPHS 36
3.1 Complexity of Graphs with Wheel Graph as their Blocks 36
3.1.1 Complexity of Wheel Graph (W_n) 36
3.1.2 Complexity of 2-Wheel Graph $(W_{n,2})$ 44
3.1.3 Complexity of m-Wheel Graph $(W_{n,m})$ 50

3.2 Complexity of Graphs with Fan Graph as their Blocks 52
3.2.1 Complexity of Fan Graph (F_n) 52
3.2.2 Complexity of Octopus Graph (O_n) 58
3.2.3 Complexity of Dragonfly Graph (Dg_n) 62
3.2.4 Complexity of Generalized Dragonfly Graph $\left(Dg_n^{(3,3)}\right)$ 65
CHAPTER IV CONCLUSIONS 69
REFERENCES 70

LIST OF FIGURES

Figure 1.1 Flowchart for Determining the Complexity of Wheel Graph W_n 5
Figure 2.1 (a) Null Graph N_5 and (b) Graph G_1 6
Figure 2.2 (a) Graph G_2 and (b) Graph H 8
Figure 2.3 (a) Graph G_3, a tree, and (b) Graph G_4, not a tree 8
Figure 2.4 (a) Tree T_1, (b) Tree T_2, (c) Tree T_3, and (d) Tree T_4 9
Figure 2.5 (a) Cycle Graph C_n and (b) Path Graph P_n 10
Figure 2.6 (a) Wheel Graph W_n and (b) m-Wheel Graph $W_{n,m}$ 10
Figure 2.7 Fan Graph F_n 11
Figure 2.8 Octopus Graph O_n 11
Figure 2.9 Dragonfly Graph Dg_n 12
Figure 2.10 Generalized Dragonfly Graph $Dg_n^{(3,3)}$ 12
Figure 2.11 Graph G_5 and its blocks B_1, B_2, B_3, and B_4 13
Figure 2.12 Graph G_6 with 5 vertices and 6 edges 23
Figure 2.13 Graph G_7 30
Figure 2.14 Graph O_3 34
Figure 3.1 Graph W_3 with $V(W_3) = 4$ and $E(W_3) = 6$ 36
Figure 3.2 (a)-(p) The spanning trees of W_3 37
Figure 3.3 Graph W_4 with $|V(W_4)| = 5$ and $|E(W_4)| = 8$ 37
Figure 3.4 Graph W_5 with $|V(W_5)| = 6$ and $|E(W_5)| = 10$ 38
Figure 3.5 Graph W_6 with $|V(W_6)| = 7$ and $|E(W_6)| = 12$ 39
Figure 3.6 Graph W_7 with $|V(W_7)| = 8$ and $|E(W_7)| = 14$ 40
Figure 3.7 Graph W_8 with $|V(W_8)| = 9$ and $|E(W_8)| = 16$ 41
Figure 3.8 Graph $W_{3,2}$ with $|V(W_{3,2})| = 7$ and $|E(W_{3,2})| = 12$ 44
Figure 3.9 Graph $W_{4,2}$ with $|V(\mathrm{W}_{4,2})| = 9$ and $|E(W_{4,2})| = 16$ 45
Figure 3.10 Graph $W_{5,2}$ with $|V(\mathrm{W}_{5,2})| = 11$ and $|E(W_{5,2})| = 20$ 47
Figure 3.11 Graph F_2 with $|V(F_2)| = 3$ and $|E(F_2)| = 3$ 52
Figure 3.12 Graph F_3 with $|V(F_3)| = 4$ and $|E(F_3)| = 5$ 53
Figure 3.13 Graph F_4 with $|V(F_4)| = 5$ and $|E(F_4)| = 7$ 54
Figure 3.14 Graph F_5 with $|V(F_5)| = 6$ and $|E(F_5)| = 9$ 55
Figure 3.15 Graph F_6 with $|V(F_6)| = 7$ and $|E(F_6)| = 11$ 55

LIST OF TABLES

Table 3.1 Complexity of graph O_n for $n = 2,3,4,5$ 59
Table 3.2 Complexity of graph Dg_n for $n = 1,2,3,4$ 62
Table 3.3 Complexity of graph $Dg_n^{(3,3)}$ for $n = 1,2,3,4$ 65

LIST OF SYMBOLS

$G = (V, E)$	: Graph G with vertex set $V(G)$ dan edge set $E(G)$
$\|V(G)\|$	: Number of vertices of graph G
$\|E(G)\|$	: Number of edges of graph G
$\deg(v)$	: Degree of a vertex v
$A(G)$	: Adjacency matrix of graph G
$D(G)$	: Degree matrix of graph G
$L(G)$	: Laplacian matrix of graph G
$\det(A)$	: Determinant of matrix A
a_{ij}	: Entry of matrix A at row i and column j
$\det\big(L(G)(i\|j)\big)$	: Determinant of the submatrix obtained by deleting the i-th row and j-th column from matrix $L(G)$
C_{ij}	: Cofactor of the entry a_{ij}
$C_{ij}\big(L(G)\big)$	: Cofactor of matrix $L(G)$
W_n	: Wheel graph with $n \in \mathbb{N}$, $n \geq 3$
$W_{n,m}$	: m-Wheel graph with $n, m \in \mathbb{N}$, $n \geq 3$
F_n	: Fan graph with $n \in \mathbb{N}$, $n \geq 2$
O_n	: Octopus graph with $n \in \mathbb{N}$, $n \geq 2$
Dg_n	: Dragonfly graph with $n \in \mathbb{N}$
$Dg_n^{(3,3)}$	: Generalized dragonfly graph with $n \in \mathbb{N}$
$\tau(G)$	: Complexity of graph G (the number of spanning trees of G)

CHAPTER I
INTRODUCTION

1.1 Research Background

A tree is one of the fundamental structures in graph theory that represents the relationships between objects in a structured manner. As an undirected connected graph without cycles, a tree consists of vertices and edges that connect them. Trees are widely used in various fields such as computing, networking, and data, due to their simple yet highly flexible nature for representing structures or relationships. An important concept within trees is the spanning tree. A spanning tree of an undirected connected graph G is defined as a subgraph of G that includes all its vertices without forming any cycles (Munir, 2010).

The complexity of a graph G is defined as the number of spanning trees in G, denoted by $\tau(G)$. The problem of counting the number of spanning trees began in 1847 when Gustav Kirchhoff discovered the relationship between the number of spanning trees in G and the cofactor of the Laplacian matrix of G (Daoud & Saleh, 2020). The complexity of graph is one of the topics in graph theory that has wide-ranging applications. It can be applied in various fields, such as investigating possible transitions of maser particles using energy analysis, calculating certain chemical isomers, and determining the number of Eulerian trails in a graph (Daoud, 2017). Furthermore, complexity has many connections with networks, as the quality and integrity of a network improve, the greater the number of spanning trees covering the network. This leads to higher possibilities for connection between any two vertices, and this ensures good rigidity and reliability (Deen & Aboamer, 2021).

According to Ocansey (2011), the matrix tree theorem can be used to determine the complexity of a graph. The matrix used in this theorem is the Laplacian matrix $L(G)$, which is one of the matrix representations of a graph. Building upon this framework, this research focuses on graphs with wheel graph and fan graph as their blocks, as the block-based approach to graph characterization has been explored only to a limited extent in existing studies. A block of a graph is

defined as a maximal nonseparable subgraph that cannot be separated further because there is no vertex whose removal would disconnect the graph (Harary, 1969). Examples of graphs with wheel graph and fan graph as their blocks are the m-wheel graph, the octopus graph, the dragonfly graph, and the generalized dragonfly graph. These graph examples form the primary focus of this research, which discusses how to formulate the complexity of graphs with wheel graph and fan graph as their blocks.

1.2 Problem Statements

Based on the background presented, the problem to be discussed in this research is how to formulate the complexity formulas for the wheel graph W_n, the fan graph F_n, and graphs whose blocks are wheel graph W_n and fan graph F_n, such as the m-wheel graph $W_{n,m}$, the octopus graph O_n, the dragonfly graph Dg_n, and the generalized dragonfly graph $Dg_n^{(3,3)}$.

1.3 Research Aims

Based on the problems outlined, the aims of this research are:

1. Formulate the complexity formulas of the wheel graph W_n and the m-wheel graph $W_{n,m}$.
2. Formulate the complexity formulas of the fan graph F_n, the octopus graph O_n, the dragonfly graph Dg_n and the generalized dragonfly graph $Dg_n^{(3,3)}$.

1.4 The Scope of the Study

In this study, the problem discussed is limited to simple, connected, and undirected graphs, specifically the wheel graph W_n, the m-wheel graph $W_{n,m}$, the fan graph F_n, the octopus graph O_n, the dragonfly graph Dg_n, and the generalized dragonfly graph $Dg_n^{(3,3)}$.

1.5 Literature Review

This research examines the complexity of graphs through a literature review. Among the various books or articles studied, several have become essential foundations for this research. One of these is the book by Munir (2010), which discussed graph theory, particularly related to trees and spanning trees.

Several previous studies have established formulas for the complexity of various types of graphs. Ocansey (2011) discussed the matrix tree theorem and the complexity of the complete bipartite graph, the complete k-partite graph, and the circulant graph. In the study, the following results were obtained:

(i) The complexity of the complete bipartite graph is:

$$\tau(K_{m,n}) = m^{n-1}n^{m-1}$$

(ii) The complexity of the complete k-partite graph is:

$$\tau(K_{n_1,n_2,\cdots,n_k}) = n^{k-2}\prod_{i=1}^{k}(n-n_i)^{n_1-1}$$

(iii) The complexity of the circulant graph is:

$$\tau[C_n(n_1,n_2,,\dots,n_k)] = \frac{1}{n}\prod_{j=1}^{n-1}\left[\sum_{m=1}^{n} a_m \varsigma^{(m-1)j}\right]$$

Sulaeman (2021) discussed the complexity of the crown graph S_n^0 and the rook graph B_{nn} using the matrix tree theorem. In the study, it was obtained that $\tau(S_n^0) = \frac{n-1}{n} \times (n^2-2n)^{n-1}$ for $n \geq 3$ and $\tau(B_{nn}) = 2^{(n-1)^2} \times n^{n^2-3}$ for $n \geq 2$.

Further study by Wei et al (2023) discusses the complexity of the general wheel graphs. In the study, the results obtained include the complexity of wheel graphs with double vertices and edges, as well as the complexity of the double-wheel graph DW. The double-wheel graph DW is defined as the graph obtained by joining the two vertices that adjacent to every vertex in each of the wheel graphs W_1 and W_2. The result of the study establishes that, if DW is a double-wheel graph basing on the wheel graphs W_1 and W_2, then its complexity is $\tau(DW) = \tau(W_1) \times \tau(W_2)$.

1.6 Research Methodology

The method used in this research is a literature review, which involves reading literature related to the complexity of graphs. Literature sources include books, journals article, and internet searches.

In general, the steps to determine the complexity of the graphs W_n, $W_{n,m}$, F_n, O_n, Dg_n, dan $Dg_n^{(3,3)}$ are as follows:

1. Determine the adjacency matrix $A(G)$ and the degree matrix $D(G)$ of the graph G, where G denotes the graphs W_n, $W_{n,m}$, F_n, O_n, Dg_n, and $Dg_n^{(3,3)}$.
2. Find the Laplacian matrix of the graph G using $L(G) = D(G) - A(G)$.
3. Compute the number of spanning trees of graph G using $\tau(G) = C_{ij}\big(L(G)\big) = (-1)^{i+j} \det\big(L(G)(i|j)\big)$. where $C_{ij}\big(L(G)\big)$ denotes the cofactor of the matrix $L(G)$ and $L(G)(i|j)$ denotes the submatrix formed by deleting the i-th row and j-th column of the matrix $L(G)$.
4. Construct the complexity formula of graph G.
5. Prove the validity of the complexity formula that has been formulated.

In detail, the flowchart for determining the complexity of the wheel graph W_n is shown in Figure 1.1. The procedure for determining the complexity of the graphs $W_{n,m}$, F_n, O_n, Dg_n, dan $Dg_n^{(3,3)}$ is generally similar to the steps for determining the complexity of the wheel graph W_n.

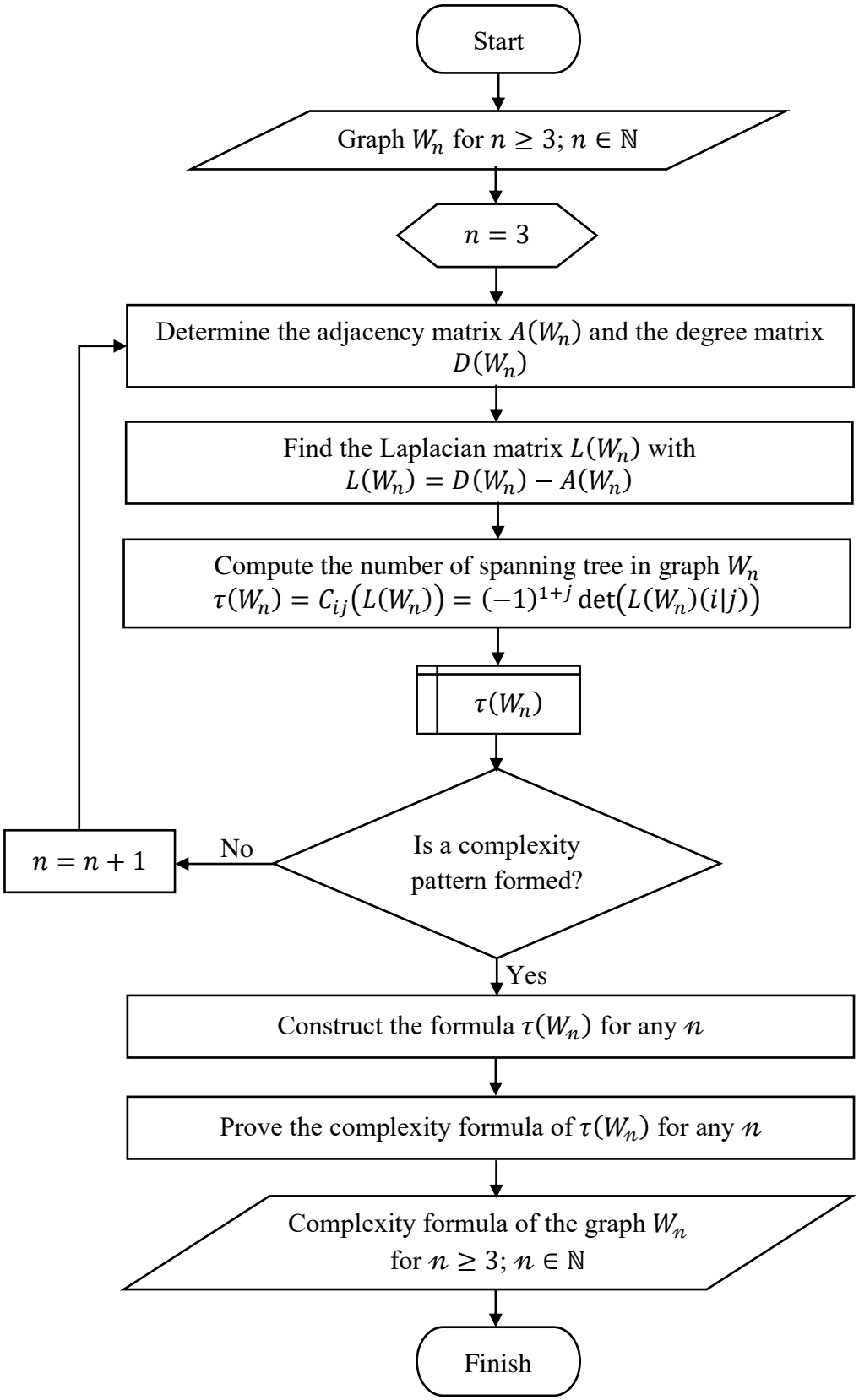


Figure 1.1 Flowchart for Determining the Complexity of Wheel Graph W_n